Partial generalizations of some Conjectures in locally symmetric Lorentz spaces

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Abstract : In this paper, first we give a notion for linear Weingarten spacelike hypersurfaces M^n with $R = aH + b_1$ in a locally symmetric Lorentz space L_1^{n+1}, where R and H are the normalized scalar curvature and the mean curvature of M^n, respectively. Furthermore, we study complete or compact linear Weingarten spacelike hypersurfaces in locally symmetric Lorentz spaces L_1^{n+1} satisfying some curvature conditions. By modifying Cheng-Yau's operator $\square$ given in [7], we introduce a modified operator L and give new estimates of $L(nH)$ and $\square(nH)$ of such spacelike hypersurfaces. Finally, we give partial generalizations of some Conjectures in locally symmetric Lorentz spaces L_1^{n+1}.

Keywords : Linear Weingarten spacelike hypersurfaces; Locally symmetric Lorentz spaces; Scalar curvature; Second fundamental form

1 introduction

Let L_p^{n+p} be an $(n+p)$-dimensional connected semi-Riemannian manifold of index p ($\geqslant 0$). It is called a semi-definite space of index p. In particular, L_1^{n+1} is called a Lorentz space. A hypersurface M^n of a Lorentz space is said to be spacelike if the metric on M^n induced from that of the Lorentz space is positive definite. When the Lorentz space is of constant curvature c, we call it Lorentz space form, denote by $\overline{M}_1^{n+1}(c)$. When $c > 0$, $\overline{M}_1^{n+1}(c) = \mathbb{S}_1^{n+1}(c)$ is called an $(n+1)$-dimensional de Sitter space; when $c = 0$, $\overline{M}_1^{n+1}(c) = \mathbb{L}_1^{n+1}(c)$ is called an $(n+1)$-dimensional Lorentz-Minkowski space; when $c < 0$, $\overline{M}_1^{n+1}(c) = \mathbb{H}_1^{n+1}(c)$ is called an $(n+1)$-dimensional anti-de Sitter space.

In 1981, it was pointed out by S. Stumbles [20] that spacelike hypersurfaces with constant mean curvature in arbitrary spacetime come from its relevance in general relativity. In fact, constant mean curvature hypersurfaces are relevant for studying propagation of gravitational radiation. Hence, many geometers studies the complete spacelike hypersurfaces with constant mean curvature H in Lorentz space forms $\overline{M}_1^{n+1}(c)$. For instance, A.J. Goddard [8] proposed the following Conjecture:

Conjecture 1. If M^n is a complete spacelike hypersurface of de Sitter space $\mathbb{S}_1^{n+1}(c)$ with constant mean curvature H, then is M^n totally umbilical ?

J. Ramanathan [19] proved Goddard's conjecture for $\mathbb{S}_1^3(1)$ and $0 \leqslant H \leqslant 1$. Moreover, when

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$H > 1$, he also showed that the conjecture is false. When $H^2 \leqslant c$ if $n = 2$ or when $n^2H^2 < 4(n-1)c$ if $n \geqslant 3$, K. Akutagawa [1] proved that Goddard's conjecture is true. S. Montiel [13] solved Goddard's problem without restriction over the range of H provided that M^n is compact. There are also many results such as [10, 15].

On the other hand, concerning the study of spacelike hypersurfaces with constant scalar curvature in a de Sitter space, H. Li [11] proposed the following problems:

Conjecture 2. If $M^n (n \geqslant 3)$ is a complete spacelike hypersurface in de Sitter space $\mathbb{S}_1^{n+1}(1)$ with constant normalized scalar curvature R satisfying $\frac{n-2}{n} \leqslant R \leqslant 1$, then is M^n totally umbilical ?

Conjecture 3. If M^n is an n-dimensional compact spacelike hypersurface in de Sitter space $\mathbb{S}_1^{n+1}(1)$ with constant scalar curvature, then is M^n totally umbilical ?

In 1997, H. Li [11] partially proved Conjecture 3 in de Sitter spaces $\mathbb{S}_1^{n+1}(c)$ and obtained Theorem 1.9([11, Theorem 4.3]).

Recently, F.E.C. Camargo et al. [4] proved that Conjecture 2 is true if the mean curvature H is bounded. There are also many results such as [3, 5] and [9].

It is natural to study complete or compact spacelike hypersurfaces with constant mean curvature or constant scalar curvature in the more general Lorentz spaces. In 2004, J. Ok Baek, Q.M. Cheng and Y. Jin Suh [16] studied the complete spacelike hypersurfaces with constant mean curvature H and gave some rigidity theorems in locally symmetric Lorentz spaces L_1^{n+1}. Recently, J.C. Liu and Z.Y. Sun [12] studied the complete spacelike hypersurfaces with constant normalized scalar curvature R and obtained some rigidity theorems in locally symmetric Lorentz spaces L_1^{n+1}.

In this paper, firstly, we recall that Choi et al. [6, 16, 21] introduced the class of $(n+1)$-dimensional Lorentz spaces L_1^{n+1} of index 1 which satisfy the following conditions for some constants c_1 and c_2:

(i) for any spacelike vector u and any timelike vector v

$$K(u, v) = -\frac{c_1}{n}, \tag{1.1}$$

(ii) for any spacelike vectors u and v

$$K(u, v) \geqslant c_2, \tag{1.2}$$

where K denotes the sectional curvature on L_1^{n+1}.

When L_1^{n+1} satisfies conditions (1.1) and (1.2), we will say that L_1^{n+1} satisfies condition $(*)$.

Remark 1.1. It is obvious that the Lorentz space form $\overline{M}_1^{n+1}(c)$ satisfies condition $(*)$ for $-\frac{c_1}{n} = c_2 = c$.

There are several examples of Lorentz spaces which are not Lorentz space forms and satisfy condition $(*)$. For instance, semi-Riemannian product manifold $H_1^k(-\frac{c_1}{n}) \times N^{n+1-k}(c_2), c_1 > 0$, and $\mathbf{R}_1^k \times S^{n+1-k}(1)$. In particular, $\mathbf{R}_1^1 \times S^n(1)$ is a so-called *Einstein Static Universe*. Also the *Robertson-Walker* spacetime $N(c, f) = I \times_f N^3(c)$ is another general example of Lorentz space, where I denotes an open interval of $\mathbf{R}_1^1$ and $f > 0$ a smooth function defined on the interval I, $N^3(c)$ a 3-dimensional Riemannian manifold of constant curvature c. $N(c, f)$ also satisfies $(*)$ if we choose an appropriate function f. For more details, we refer the readers to [6, 16, 21]

In order to present our main theorems, we will introduce some basic facts and notations. Let $\overline{R}_{CD}$ be the components of the Ricci tensor of L_1^{n+1} satisfying $(*)$, then the scalar curvature $\overline{R}$ of

L_1^{n+1} is given by

$$\overline{R} = \sum_{A=1}^{n+1} \epsilon_A \overline{R}_{AA} = -2\sum_{i=1}^{n} \overline{R}_{(n+1)ii(n+1)} + \sum_{i,j=1}^{n} \overline{R}_{ijji} = 2c_1 + \sum_{i,j=1}^{n} \overline{R}_{ijji}.$$

It is well known that $\overline{R}$ is constant when the Lorentz space L_1^{n+1} is locally symmetric, so $\sum_{i,j=1}^{n} \overline{R}_{ijji}$ is constant. From (2.3) in Section 2, we can define a P such that

$$n(n-1)P = n^2H^2 - S = \sum_{i,j=1}^{n} \overline{R}_{ijji} - n(n-1)R. \tag{1.3}$$

Hence, when M^n is a spacelike hypersurface in locally symmetric Lorentz spaces L_1^{n+1} satisfying $(*)$, we conclude from (1.3) that the normalized scalar curvature R of M^n is constant if and only if P is constant.

Next we will introduce a notion for linear Weingarten spacelike hypersurfaces in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$ as follows:

Definition 1.2. Let M^n be a spacelike hypersurface in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$. We call M^n a *linear Weingarten spacelike hypersurface* if the normalized scalar curvature R and the mean curvature H of M^n satisfy the following conditions: $eR = aH + b_1$, where e, a, b_1 are constants and $e^2 + a^2 \neq 0$.

Remark 1.3. Let $e = 0$ and $a \neq 0$ in Definition 1.2, a linear Weingarten spacelike hypersurface M^n reduces to a spacelike hypersurface with constant mean curvature H. Let $a = 0$ and $e \neq 0$ in Definition 1.2, a linear Weingarten spacelike hypersurface M^n reduces to a spacelike hypersurface with constant normalized scalar curvature R. Hence, the linear Weingarten spacelike hypersurfaces can be regarded as a natural generalization of spacelike hypersurfaces with constant mean curvature H or with constant normalized scalar curvature R in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$.

In 2010, J.C. Liu and Z.Y. Sun [12] gave partial generalizations of Conjecture 2 and [4, Theorem 1.2] in locally symmetric Lorentz spaces L_1^{n+1} satisfying $(*)$ and obtained the following result.

Theorem 1.4([12]). *Let $M^n(n \geqslant 3)$ be a complete spacelike hypersurface with constant normalized scalar curvature R in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$. Suppose that M^n has bounded mean curvature H. If the constant P defined by (1.3) satisfies $0 \leqslant P \leqslant \frac{2c}{n}$ and $c = 2c_2 + \frac{c_1}{n} > 0$, where c_1, c_2 are given as in $(*)$, then M^n is totally umbilical.*

In 2008, F.E.C. Camargo, R.M.B. Chaves and L.A.M. Sousa Jr.[4] studied the complete spacelike hypersurfaces with constant normalized scalar curvature R in de Sitter spaces $\mathbb{S}_1^{n+1}(c)$ and proved the following result.

Theorem 1.5([4]). *Let $M^n(n \geqslant 3)$ be a complete spacelike hypersurface with constant normalized scalar curvature R in a de Sitter space $\mathbb{S}_1^{n+1}(c)$. If the squared length S of the second fundamental form of M^n satisfies*

$$\sup S < 2\sqrt{n-1}c$$

and $R \leqslant c$, then M^n is totally umbilical.

In Section 3, by modifying Cheng-Yau's operator $\square$ given in [7], we study complete linear Weingarten spacelike hypersurfaces in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$ and give generalizations of Theorem 1.4([12, Theorem 1.2(i)]) and Theorem 1.5([4, Theorem 1.2]). Thus, we get Theorems 1.6 and 1.8.

Theorem 1.6. *Let $M^n(n \geqslant 3)$ be a complete spacelike hypersurface in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$. Suppose that M^n has bounded mean curvature H. If the normalized scalar curvature R and the mean curvature H of M^n satisfy the following conditions: $R = aH + b_1$, $b_1 = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \overline{R}_{ijji} - b$, $(n-1)a^2 + 4nb \geqslant 0$, $a \geqslant 0$, $b \leqslant \frac{2c}{n}$ and $c = 2c_2 + \frac{c_1}{n} > 0$, where a, b, b_1 are constants and c_1, c_2 are given as in $(*)$, then M^n is totally umbilical.*

Remark 1.7. It is well known that $\overline{R}$ is constant when the Lorentz space L_1^{n+1} is locally symmetric, so $\sum_{i,j=1}^{n} \overline{R}_{ijji}$ is constant. Hence b_1 is constant. When we take $a = 0$ in Theorem 1.6, combining (1.3), we obtain that $P = b$ is constant and $0 \leqslant P \leqslant \frac{2c}{n}$. Hence, Theorem 1.6 is a generalization of Theorem 1.4. If $a = 0$ and L_1^{n+1} is the de Sitter space $\mathbb{S}_1^{n+1}(c)$ in Theorem 1.6, then $-\frac{c_1}{n} = c_2 = c$ and $0 \leqslant b = c - R \leqslant \frac{2c}{n}$ following from (1.3). At the same time, $0 \leqslant b = c - R \leqslant \frac{2c}{n}$ becomes $\frac{n-2}{n}c \leqslant R \leqslant c$ and R is constant. Hence, Theorem 1.6 is also a generalization of the result due to F.E.C. Camargo et al. in [4], saying that a complete spacelike hypersurface M^n $(n \geqslant 3)$ in a de Sitter space $\mathbb{S}_1^{n+1}(c)$ with constant normalized scalar curvature R satisfying $\frac{n-2}{n}c \leqslant R \leqslant c$ must be totally umbilical provided that M^n has bounded mean curvature H.

For example, we consider the spacelike hypersurface immersed into $\mathbb{S}_1^{n+1}(1)$ defined by $T_{k,r} = \{x \in \mathbb{S}_1^{n+1}(1) | -x_0^2 + x_1^2 + \ldots + x_k^2 = -\sinh^2 r\}$, where r is a positive real number and $1 \leqslant k \leqslant n-1$. $T_{k,r}$ is complete and isometric to the Riemannian product $\mathbb{H}^k(1 - \coth^2 r) \times \mathbb{S}^{n-k}(1 - \tanh^2 r)$ of a k-dimensional hyperbolic space and an $(n-k)$-dimensional sphere of constant sectional curvatures $1 - \coth^2 r$ and $1 - \tanh^2 r$, respectively. It follows from [9] that if $k = 1$, then R satisfies $0 < R = \frac{n-2}{n}(1 - \tanh^2 r) < \frac{n-2}{n}$; similarly, if $k = n-1 \geqslant 2$, we see that $R = \frac{n-2}{n}(1 - \coth^2 r) < 0$. Thus, for any R satisfying $0 < R < \frac{n-2}{n}$ and for any $R < 0$, we can choose r such that the hypersurfaces $T_{1,r}$ and $T_{n-1,r}$, respectively, are complete, non-totally umbilical and have constant normalized scalar curvature R. Hence, when $M^n(n \geqslant 3)$ is a complete spacelike hyperusrface, the hypothesis that $0 \leqslant P \leqslant \frac{2c}{n}$ is essential to umbilicity of M^n in Theorem 1.4. Without assumption that the normalized scalar curvature R is constant in Theorem 1.6, we generalize the assumption condition $0 \leqslant P \leqslant \frac{2c}{n}$ in Theorem 1.4 to more general situations. Hence, the hypothesis that $R = aH + b_1$ is essential to umbilicity of M^n in Theorem 1.6, where $b_1 = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \overline{R}_{ijji} - b$, $(n-1)a^2 + 4nb \geqslant 0$, $a \geqslant 0$, $b \leqslant \frac{2c}{n}$ and $c = 2c_2 + \frac{c_1}{n} > 0$.

Theorem 1.8. *Let $M^n(n \geqslant 3)$ be a complete spacelike hypersurface in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$. Suppose that the squared length S of the second fundamental form of M^n satisfies $\sup S < 2\sqrt{n-1}c$, where $c = 2c_2 + \frac{c_1}{n}$ and c_1, c_2 are given as in $(*)$. If the normalized scalar curvature R and the mean curvature H of M^n satisfy the following conditions: $R = aH + b_1$, $b_1 = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \overline{R}_{ijji} - b$, $(n-1)a^2 + 4nb \geqslant 0$, and $a \geqslant 0$, where a, b and b_1 are constants. then M^n is totally umbilical.*

In 1997, H. Li [11] partially solved Conjecture 3 in de Sitter spaces $\mathbb{S}_1^{n+1}(c)$ and obtained the following result.

Theorem 1.9([11]). *Let $M^n(n \geqslant 3)$ be a compact spacelike hypersurface with constant normalized scalar curvature R in a de Sitter space $\mathbb{S}_1^{n+1}(c)$. If $\frac{n-2}{n}c \leqslant R \leqslant c$, then M^n is totally umbilical.*

In 2010, J.C. Liu and Z.Y. Sun [12] gave a generalization of Theorem 1.9 in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$ and obtained Theorem 1.10.

Theorem 1.10([12]). *Let $M^n(n \geqslant 3)$ be a compact spacelike hypersurface with constant normalized scalar curvature R in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$. If the constant P*

defined by (1.3) *satisfies* $0 \leqslant P \leqslant \frac{2c}{n}$ *and* $c = 2c_2 + \frac{c_1}{n} > 0$, *where* c_1, c_2 *are given as in* $(*)$, *then* M^n *is totally umbilical.*

In Section 4, by using Cheng-Yau's operator $\square$ given in [7], we study compact linear Weingarten spacelike hypersurfaces in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$ and give generalizations of Theorem 1.9([11, Theorme 4.3]) and Theorem 1.10([12, Theorem 1.1]). Then, we obtain Theorems 1.11 and 1.13.

Theorem 1.11. *Let* $M^n (n \geqslant 3)$ *be a compact spacelike hypersurface in a locally symmetric Lorentz space* L_1^{n+1} *satisfying* $(*)$. *If the normalized scalar curvature* R *and the mean curvature* H *of* M^n *satisfy the following conditions*: $R = aH + b_1$, $b_1 = \frac{1}{n(n-1)} \sum_{i,j=1}^n \overline{R}_{ijji} - b$, $(n-1)a^2 + 4nb \geqslant 0$, $a \geqslant 0$, $b \leqslant \frac{2c}{n}$, *and* $c = 2c_2 + \frac{c_1}{n} > 0$, *where* a, b, b_1 *are constants and* c_1, c_2 *are given as in* $(*)$, *then* M^n *is totally umbilical.*

Remark 1.12. When we take $a = 0$ in Theorem 1.11, we know that $P = b$ is constant and $0 \leqslant P = b \leqslant \frac{2c}{n}$. Thus, Theorem 1.11 generalizes Theorems 1.9 and 1.10.

Theorem 1.13. *Let* $M^n (n \geqslant 3)$ *be a compact spacelike hypersurface in a locally symmetric Lorentz space* L_1^{n+1} *satisfying* $(*)$. *Suppose that the squared length* S *of the second fundamental form of* M^n *satisfies* $S < 2\sqrt{n-1}c$, *where* $c = 2c_2 + \frac{c_1}{n}$ *and* c_1, c_2 *are given as in* $(*)$. *If the normalized scalar curvature* R *and the mean curvature* H *of* M^n *satisfy the following conditions*: $R = aH + b_1$, $b_1 = \frac{1}{n(n-1)} \sum_{i,j=1}^n \overline{R}_{ijji} - b$, $(n-1)a^2 + 4nb \geqslant 0$ *and* $a \geqslant 0$, *where* a, b *and* b_1 *are constants. then* M^n *is totally umbilical.*

Remark 1.14. In this paper, the spacelike hypersurfaces M^n in Theorems 1.6, 1.8, 1.11 and 1.13 satisfying $R = aH + b_1$ are linear Weingarten spacelike hypersurfaces in Definition 1.2.

2 Preliminaries

In this section, we will introduce some basic facts and give estimate the Laplacian ΔS of the squared length S of the second fundamental form for spacelike hypersurfaces in locally symmetric Lorentz spaces L_1^{n+1} satisfying $(*)$. We shall make use of the following convention on the ranges of indices: $1 \leqslant A, B, C, \ldots \leqslant n+1$; $1 \leqslant i, j, k, \ldots \leqslant n$.

We assume that M^n is a spacelike hypersurface in Lorentz spaces L_1^{n+1}. Choose a local field of pseudo-Riemannian orthonormal frames $\{e_1, \ldots, e_{n+1}\}$ in L_1^{n+1} such that, restricted to M^n, $\{e_1, \ldots, e_n\}$ are tangent to M^n and e_{n+1} is normal to M^n. That is, $\{e_1, \ldots, e_n\}$ are spacelike vectors and e_{n+1} is a timelike vector. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the fields of dual frames and the connection forms of L_1^{n+1}, respectively. Let $\epsilon_i = 1, \epsilon_{n+1} = -1$, then the structure equations of L_1^{n+1} are given by

$$\begin{aligned} d\omega_A &= -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D \overline{R}_{ABCD} \omega_C \wedge \omega_D. \end{aligned}$$

Here the components $\overline{R}_{CD}$ of the Ricci tensor and the scalar curvature $\overline{R}$ of Lorentz spaces L_1^{n+1} are given, respectively, by

$$\overline{R}_{CD} = \sum_B \epsilon_B \overline{R}_{BCDB}, \qquad \overline{R} = \sum_A \epsilon_A \overline{R}_{AA}.$$

The components $\overline{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor $\overline{R}$ are defined by

$$\begin{aligned}\sum_E \epsilon_E \overline{R}_{ABCD;E}\omega_E = d\overline{R}_{ABCD} - \sum_E \epsilon_E(&\overline{R}_{EBCD}\omega_{EA} \\ &+ \overline{R}_{AECD}\omega_{EB} + \overline{R}_{ABED}\omega_{EC} + \overline{R}_{ABCE}\omega_{ED}).\end{aligned}$$

We restrict these forms to M^n in L_1^{n+1}, then $\omega_{n+1} = 0$. Hence, we have $\sum_i \omega_{(n+1)i} \wedge \omega_i = 0$. Using Cartan's lemma, we know that there are h_{ij} such that $\omega_{(n+1)i} = \sum_j h_{ij}\omega_j$ and $h_{ij} = h_{ji}$, where the h_{ij} are the coefficients of the second fundamental form of M^n. This gives the second fundamental form of M^n, $h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$.

The Gauss equation, components R_{ij} of the Ricci tensor and the normalized scalar curvature R of M^n are given, respectively, by

$$R_{ijkl} = \overline{R}_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}), \tag{2.1}$$

$$R_{ij} = \sum_k \overline{R}_{kijk} - nHh_{ij} + \sum_k h_{ik}h_{kj}, \tag{2.2}$$

$$n(n-1)R = \sum_{i,j} \overline{R}_{ijji} - n^2H^2 + S, \tag{2.3}$$

where $H = \frac{1}{n}\sum_j h_{jj}$ and $S = \sum_{i,j} h_{ij}^2$ are the mean curvature and the squared length of the second fundamental form of M^n, respectively.

Let h_{ijk} and h_{ijkl} be the first and the second covariant derivatives of h_{ij}, respectively, so that

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{ik}\omega_{kj} - \sum_k h_{kj}\omega_{ki},$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ljk}\omega_{li} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}.$$

Thus, we have the Codazzi equation and the Ricci identity

$$h_{ijk} - h_{ikj} = \overline{R}_{(n+1)ijk}, \tag{2.4}$$

$$h_{ijkl} - h_{ijlk} = -\sum_m h_{im}R_{mjkl} - \sum_m h_{jm}R_{mikl}. \tag{2.5}$$

Let $\overline{R}_{ABCD;E}$ be the covariant derivative of $\overline{R}_{ABCD}$. Thus, restricted on M^n, $\overline{R}_{(n+1)ijk;l}$ is given by

$$\overline{R}_{(n+1)ijk;l} = \overline{R}_{(n+1)ijkl} + \overline{R}_{(n+1)i(n+1)k}h_{jl} + \overline{R}_{(n+1)ij(n+1)}h_{kl} + \sum_m \overline{R}_{mijk}h_{ml}, \tag{2.6}$$

where $\overline{R}_{(n+1)ijk;l}$ denotes the covariant derivative of $\overline{R}_{(n+1)ijk}$ as a tensor on M^n so that

$$\begin{aligned}\sum_l \overline{R}_{(n+1)ijk;l}\omega_l = d\overline{R}_{(n+1)ijk} &- \sum_l \overline{R}_{(n+1)ljk}\omega_{li} \\ &- \sum_l \overline{R}_{(n+1)ilk}\omega_{lj} - \sum_l \overline{R}_{(n+1)ijl}\omega_{lk}.\end{aligned}$$

Next we compute the Laplacian $\Delta h_{ij} = \sum_k h_{ijkk}$. From (2.4) and (2.5), we have

$$\begin{aligned}\Delta h_{ij} &= \sum_k h_{ikjk} + \overline{R}_{(n+1)ijk;k} \\ &= \sum_k \left(h_{kikj} - \sum_l (h_{kl}R_{lijk} + h_{il}R_{lkjk}) + \overline{R}_{(n+1)ijk;k} \right).\end{aligned}$$

From $h_{kikj} = h_{kkij} + \overline{R}_{(n+1)kik;j}$, we get

$$\Delta h_{ij} = (nH)_{ij} + \sum_k \left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\right) - \sum_{k,l}(h_{kl}R_{lijk} + h_{il}R_{lkjk}). \tag{2.7}$$

From (2.1) and (2.6) and (2.7), we obtain

$$\begin{aligned}\Delta h_{ij} =&(nH)_{ij} + \sum_k \left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\right) - \sum_k (h_{kk}\overline{R}_{(n+1)ij(n+1)} \\ &+ h_{ij}\overline{R}_{(n+1)k(n+1)k}) - \sum_{k,l}\left(2h_{kl}\overline{R}_{lijk} + h_{jl}\overline{R}_{lkik} + h_{il}\overline{R}_{lkjk}\right) \\ &- nH\sum_l h_{il}h_{lj} + Sh_{ij}.\end{aligned}$$

According to the above equation, the Laplacian ΔS of the squared length S of the second fundamental form h_{ij} of M^n is obtained

$$\begin{aligned}\frac{1}{2}\Delta S =& \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij} \\ =& \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j}(nH)_{ij}h_{ij} + \sum_{i,j,k}\left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\right)h_{ij} \\ &- \left(\sum_{i,j} nHh_{ij}\overline{R}_{(n+1)ij(n+1)} + S\sum_k \overline{R}_{(n+1)k(n+1)k}\right) \\ &- 2\sum_{i,j,k,l}(h_{kl}h_{ij}\overline{R}_{lijk} + h_{il}h_{ij}\overline{R}_{lkjk}) - nH\sum_{i,j,l} h_{il}h_{lj}h_{ij} + S^2.\end{aligned} \tag{2.8}$$

Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i\delta_{ij}$, where λ_i, $1 \leqslant i \leqslant n$, are principal curvatures of M^n. Estimating the right-hand side of (2.8) by using the curvature conditions $(*)$, the following lemma was obtained by J.C. Liu and Z.Y. Sun.

Lemma 2.1 ([12, Lemma 2.1]). *Let M^n be a spacelike hypersurface in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$, then*

$$\frac{1}{2}\Delta S \geqslant \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i(nH)_{ii} + nc(S - nH^2) + \left(S^2 - nH\sum_i \lambda_i^3\right), \tag{2.9}$$

where $c = 2c_2 + \frac{c_1}{n}$ and c_1, c_2 are given as in $(*)$.

In the following, we will continue to calculate ΔS for spacelike hypersurfaces in locally symmetry Lorentz spaces satisfying $(*)$. Thus, we need the following algebraic Lemma.

Lemma 2.2 ([2, 17]). *Let $\mu_1, \ldots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = B^2$, where $B \geqslant 0$ is constant. Then*

$$\left|\sum_i \mu_i^3\right| \leqslant \frac{n-2}{\sqrt{n(n-1)}}B^3$$

and equality holds if and only if at least $n-1$ of the μ_i's are equal.

Let $\phi = \sum_{i,j}\phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n, where $\phi_{ij} = h_{ij} - H\delta_{ij}$. It is easy to check that ϕ is traceless. Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i\delta_{ij}$ and $\phi_{ij} = \mu_i\delta_{ij}$. Let $|\phi|^2 = \sum_i \mu_i^2$. A direct computation gets

$$|\phi|^2 = S - nH^2 = \frac{1}{2n}\sum_{i,j}(\lambda_i - \lambda_j)^2. \tag{2.10}$$

Hence, $|\phi|^2 = 0$ if and only if M^n is totally umbilical. We also get

$$\sum_i \lambda_i^3 = nH^3 + 3H\sum_i \mu_i^2 + \sum_i \mu_i^3.$$

By applying Lemma 2.2 to the real numbers $\mu_1, \ldots, \mu_n$, we obtain

$$\begin{aligned} -nH\sum_i \lambda_i^3 =& -n^2H^4 - 3nH^2\sum_i \mu_i^2 - nH\sum_i \mu_i^3 \\ \geqslant& 2n^2H^4 - 3nSH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|(S-nH^2)^{\frac{3}{2}}. \end{aligned} \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9), we obtain the following lemma.

Lemma 2.3. *Let M^n be a spacelike hypersurface in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$, then*

$$\frac{1}{2}\Delta S \geqslant \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + |\phi|^2 L_{|H|}(|\phi|), \tag{2.12}$$

where $|\phi|^2 = S - nH^2$, $L_{|H|}(|\phi|) = |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + nc - nH^2$, $c = 2c_2 + \frac{c_1}{n}$ and c_1, c_2 are given as in $(*)$.

3 Complete linear Weingarten spacelike hypersurfaces in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$

In this section, according to Cheng and Yau $\square$ given in [7], first we introduce a modified operator L acting on any C^2-function f by

$$L(f) = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij} + \frac{(n-1)a}{2}\Delta f, \tag{3.1}$$

where a is constant.

Cheng-Yau [7] gave a lower estimate of $\sum_{i,j,k} h_{ijk}^2$ which is very important in the proof of their theorem. They proved that, for a hypersurface in a space form of constant sectional curvature c, if the normalized scalar curvature R is constant and $R \geqslant c$, then $\sum_{i,j,k} h_{ijk}^2 \geqslant n^2|\nabla H|^2$, where h_{ijk}'s are components of the covariant differentiation of the second fundamental form.

For the spacelike hypersurfaces M^n in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$, without assumption that the normalized scalar curvature R of M^n is constant, we also obtain the estimate $\sum_{i,j,k} h_{ijk}^2 \geqslant n^2|\nabla H|^2$ in the proof of Proposition 3.1.

Next we will prove Propositions 3.1 and 3.3 which will play a crucial role in the proofs of Theorems 1.6 and 1.8.

Proposition 3.1. *Let $M^n (n \geqslant 3)$ be a spacelike hypersurface in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$. If the normalized scalar curvature R and the mean curvature H of M^n satisfy the following conditions: $R = aH + b_1$, $b_1 = \frac{1}{n(n-1)}\sum_{i,j=1}^n \overline{R}_{ijji} - b$ and $(n-1)a^2 + 4nb \geqslant 0$, where a, b and b_1 are constants. then*

$$L(nH) \geqslant |\phi|^2 L_{|H|}(|\phi|), \tag{3.2}$$

where $|\phi|^2 = S - nH^2$, $L_{|H|}(|\phi|) = |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + nc - nH^2$, $c = 2c_2 + \frac{c_1}{n} > 0$ and c_1, c_2 are given as in $(*)$.

Proof. Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$. Since $R = aH + b_1$, it follows from (2.3) that

$$n^2H^2 - S = \sum_{i,j=1}^{n} \overline{R}_{ijji} - n(n-1)R = -n(n-1)(aH - b). \tag{3.3}$$

Noticing that $nH\Delta(nH) = \frac{1}{2}\Delta(nH)^2 - n^2|\nabla H|^2$, it follows from (3.1) and (3.3) that

$$\begin{aligned} L(nH) =& \sum_{i,j}(nH\delta_{ij} - h_{ij})(nH)_{ij} + \frac{(n-1)a}{2}\Delta(nH) \\ =& nH\Delta(nH) - \sum_{i}\lambda_i(nH)_{ii} + \frac{1}{2}\Delta\left[S - n^2H^2 + n(n-1)b\right] \\ =& \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_{i}\lambda_i(nH)_{ii}. \end{aligned} \tag{3.4}$$

Thus, it follows from (2.12) and (3.4) that

$$L(nH) \geqslant \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + |\phi|^2 L_{|H|}(|\phi|), \tag{3.5}$$

where $|\phi|^2 = S - nH^2$ and $L_{|H|}(|\phi|) = |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + nc - nH^2$.

Differentiating formula (3.3) exteriorly yields $2\sum_{i,j} h_{ij}h_{ijk} = 2n^2HH_k + n(n-1)aH_k$, then by using Cauchy-Schwarz inequality we have

$$4S\sum_{i,j,k} h_{ijk}^2 \geqslant 4\sum_{k}\left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 = \left[2n^2H + n(n-1)a\right]^2|\nabla H|^2. \tag{3.6}$$

Combining $(n-1)a^2 + 4nb \geqslant 0$ and (3.3), we have

$$\begin{aligned} \left[2n^2H + n(n-1)a\right]^2 - 4n^2S &= 4n^4H^2 + 4n^3(n-1)aH + n^2(n-1)^2a^2 \\ &\quad - 4n^2\left[n^2H^2 + n(n-1)(aH - b)\right] \\ &= n^2(n-1)\left[(n-1)a^2 + 4nb\right] \\ &\geqslant 0. \end{aligned} \tag{3.7}$$

Thus, we conclude from (3.6) and (3.7) that

$$\sum_{i,j,k} h_{ijk}^2 \geqslant n^2|\nabla H|^2. \tag{3.8}$$

Consequently, (3.2) follows from (3.5) and (3.8). Finally, Proposition 3.1 is proved. □

We also need the following lemma in the proof of Proposition 3.3.

Lemma 3.2 ([18]). *Let M^n be an n-dimensional complete Riemannion manifold whose sectional curvature is bounded from below and $F : M^n \to \mathbb{R}$ be a smooth function which is bounded from above on M^n. Then there exists a sequence of points $\{x_k\} \in M^n$ such that*

$$\begin{aligned} &\lim_{k\to\infty} F(x_k) = \sup F, \\ &\lim_{k\to\infty} |\nabla F(x_k)| = 0, \\ &\lim_{k\to\infty} \sup\max\{\left(\nabla^2 F(x_k)\right)(X, X) : \ |X| = 1\} \leqslant 0. \end{aligned}$$

Proposition 3.3. *Let $M^n (n \geqslant 3)$ be a complete spacelike hypersurface in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$. Suppose that M^n has bounded mean curvature H. If the normalized scalar curvature R and the mean curvature H of M^n satisfy the following conditions: $R = aH + b_1$, $b_1 = \frac{1}{n(n-1)} \sum_{i,j=1}^n \overline{R}_{ijji} - b$, $(n-1)a^2 + 4nb \geqslant 0$ and $a \geqslant 0$, where a, b and b_1 are constants, then there is a sequence of points $\{x_k\} \in M^n$ such that*

$$\begin{aligned} &\lim_{k\to\infty} nH(x_k) = \sup(nH), \\ &\lim_{k\to\infty} |\nabla(nH)(x_k)| = 0, \\ &\lim_{k\to\infty} \sup\left(L(nH)(x_k)\right) \leqslant 0. \end{aligned} \tag{3.9}$$

Proof. Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$. If $H \equiv 0$, the proposition is obvious. Let us suppose that H is not identically zero. By changing the orientation of M^n if necessary, we may assume $\sup H > 0$. In view of (3.1), $L(nH)$ is given by

$$L(nH) = \sum_i (nH - \lambda_i)(nH)_{ii} + \frac{(n-1)a}{2} \sum_i (nH)_{ii}. \tag{3.10}$$

Since $(n-1)a^2 + 4nb \geqslant 0$, it follows from (3.3) that

$$\begin{aligned} (\lambda_i)^2 \leqslant S =& n^2H^2 + n(n-1)(aH - b) \\ =& \left[nH + \frac{(n-1)a}{2}\right]^2 - \frac{(n-1)^2a^2}{4} - n(n-1)b \\ \leqslant& \left[nH + \frac{(n-1)a}{2}\right]^2. \end{aligned} \tag{3.11}$$

Thus, it follows from (3.11) that

$$|\lambda_i| \leqslant \left| nH + \frac{(n-1)a}{2} \right|. \tag{3.12}$$

From (1.2) and (2.2), we have

$$\begin{aligned} R_{ii} =& \sum_k \overline{R}_{kiik} - nHh_{ii} + \sum_k (h_{ik})^2 \\ \geqslant& \sum_k \overline{R}_{kiik} - \frac{nH}{2} 2h_{ii} + (h_{ii})^2 \\ =& \sum_k \overline{R}_{kiik} + \left(h_{ii} - \frac{nH}{2}\right)^2 - \frac{n^2H^2}{4} \\ \geqslant& nc_2 - \frac{n^2H^2}{4}. \end{aligned} \tag{3.13}$$

Since H is bounded, it follows from (3.13) that the sectional curvatures of M^n are bounded from below. Therefore, we may apply Lemma 3.2 to the function nH, obtaining a sequence of points $\{x_k\} \in M^n$ such that

$$\lim_{k\to\infty} nH(x_k) = \sup(nH), \quad \lim_{k\to\infty} |\nabla(nH)(x_k)| = 0, \quad \lim_{k\to\infty} \sup\left(nH_{ii}(x_k)\right) \leqslant 0. \tag{3.14}$$

Since H is bounded, taking subsequences if necessary, we can obtain a sequence of points $\{x_k\} \in M^n$ which satisfies (3.14) and such that $H(x_k) \geqslant 0$ (by changing the orientation of M^n if necessary).

Since $a \geqslant 0$, it follows from (3.12) that

$$\begin{aligned}0 \leqslant nH(x_k) + \frac{(n-1)a}{2} - |\lambda_i(x_k)| \leqslant & nH(x_k) + \frac{(n-1)a}{2} - \lambda_i(x_k) \\ \leqslant & nH(x_k) + \frac{(n-1)a}{2} + |\lambda_i(x_k)| \\ \leqslant & 2\left[nH(x_k) + \frac{(n-1)a}{2}\right].\end{aligned} \tag{3.15}$$

Using once more the fact that H is bounded, we can conclude from (3.15) that $\{nH(x_k) + \frac{(n-1)a}{2} - \lambda_i(x_k)\}$ is non-negative and bounded. By applying $L(nH)$ at x_k, taking the limit and using (3.14) and (3.15), we obtain

$$\begin{aligned}\lim_{k\to\infty} \sup\,(L(nH)(x_k)) \leqslant & \sum_i \lim_{k\to\infty} \sup \left(nH + \frac{(n-1)a}{2} - \lambda_i\right)(x_k) nH_{ii}(x_k) \\ \leqslant & \ 0.\end{aligned}$$

Finally, Proposition 3.3 is proved. □

Proof of Theorem 1.6. If M^n is maximal, i.e., $H \equiv 0$, according to Nishikawa's result [14], we know that M^n is totally geodesic. We can assume that H is not identically zero. Hence, by Proposition 3.3 we can obtain a sequence of points $\{x_k\} \in M^n$ such that

$$\lim_{k\to\infty} \sup\,(L(nH)(x_k)) \leqslant 0, \qquad \lim_{k\to\infty} (nH)(x_k) = \sup(nH) > 0. \tag{3.16}$$

From (2.10) and (3.3), we have

$$|\phi|^2 = n(n-1)(H^2 + aH - b). \tag{3.17}$$

In view of $\lim_{k\to\infty}(nH)(x_k) = \sup(nH) > 0$ and $a \geqslant 0$, it follows from (3.17) that

$$\lim_{k\to\infty} |\phi|^2(x_k) = \sup|\phi|^2. \tag{3.18}$$

Next, we will consider the following polynomial given by

$$L_{\sup|H|}(x) = x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} \sup|H| x + nc - n\sup H^2.$$

We claim that

$$L_{\sup|H|}(\sup|\phi|) > 0. \tag{3.19}$$

Indeed, if $\sup H^2 < \frac{4(n-1)}{n^2}c$, then the discriminant of $L_{\sup|H|}(x)$ is negative. Therefore, we have $L_{\sup|H|}(\sup|\phi|) > 0$. Suppose that $\sup H^2 \geqslant \frac{4(n-1)}{n^2}c$. Let ξ be the biggest root of the equation $L_{\sup|H|}(x) = 0$, which is positive. We know that ξ is the only one root of $L_{\sup|H|}(x)$ if $\sup H^2 = \frac{4(n-1)}{n^2}c$.

If we can prove that $(\sup|\phi|)^2 = \sup|\phi|^2 > \xi^2$, then we have $\sup|\phi| > \xi$. Hence, $L_{\sup|H|}(\sup|\phi|) > 0$. Since $a \geqslant 0$, $b \leqslant \frac{2c}{n}$ and $c > 0$, it follows from (3.17) that

$$\sup|\phi|^2 = n(n-1)(\sup H^2 + a\sup H - b) \geqslant (n-1)(n\sup H^2 - 2c). \tag{3.20}$$

By virtue of (3.20), it is straightforward to verify that

$$\begin{aligned}&\sup|\phi|^2 - \xi^2 \\ &\geqslant \frac{n-2}{2(n-1)}\left[n^2\sup H^2 - n\sup H\sqrt{n^2\sup H^2 - 4(n-1)c} - 2(n-1)c\right].\end{aligned}$$

Thus, $\sup|\phi|^2-\xi^2>0$ if and only if

$$n^2\sup H^2-n\sup H\sqrt{n^2\sup H^2-4(n-1)c}-2(n-1)c>0. \tag{3.21}$$

Since $n\sup H>0$ in (3.16) and $\sup H^2\geqslant\frac{4(n-1)}{n^2}c$, we have

$$\begin{aligned}&n^2\sup H^2-n\sup H\sqrt{n^2\sup H^2-4(n-1)c}-2(n-1)c>0\\ \Leftrightarrow& n^2\sup H^2-2(n-1)c>n\sup H\sqrt{n^2\sup H^2-4(n-1)c}\\ \Leftrightarrow&(n^2\sup H^2-2(n-1)c)^2>n^2\sup H^2(n^2\sup H^2-4(n-1)c)\\ \Leftrightarrow&4(n-1)^2c^2>0.\end{aligned}$$

Hence the inequality (3.21) is equivalent to $4(n-1)^2c^2>0$, which is true because of $c>0$. Hence, $\sup|\phi|^2-\xi^2>0$, which proves our claim.

Evaluating (3.2) at the points x_k of the sequence, taking the limit and using (3.16) and (3.18), we obtain that

$$\begin{aligned}0\geqslant&\lim_{k\to\infty}\sup\left(L(nH)(x_k)\right)\\ \geqslant&\sup|\phi|^2\left(\sup|\phi|^2-\frac{n(n-2)}{\sqrt{n(n-1)}}\sup|H|\sup|\phi|+nc-n\sup H^2\right)\\ =&\sup|\phi|^2L_{\sup|H|}(\sup|\phi|),\end{aligned} \tag{3.22}$$

where

$$L_{\sup|H|}(\sup|\phi|)=\sup|\phi|^2-\frac{n(n-2)}{\sqrt{n(n-1)}}\sup|H|\sup|\phi|+nc-n\sup H^2.$$

Therefore, we can conclude from (3.19) and (3.22) that $\sup|\phi|^2=0$. That is, $|\phi|^2=0$ which shows M^n is totally umbilical. This completes the proof of Theorem 1.6. □

If L_1^{n+1} is a de Sitter space $\mathbb{S}_1^{n+1}(c)$ in Theorem 1.6, then $-\frac{c_1}{n}=c_2=c$ and $R=aH+c-b$. Hence, we obtain the following corollary.

Corollary 3.4. *Let $M^n(n\geqslant3)$ be a complete spacelike hypersurface in a de Sitter space $\mathbb{S}_1^{n+1}(c)$. Suppose that M^n has bounded mean curvature H. If the normalized scalar curvature R and the mean curvature H of M^n satisfy the following conditions: $R=aH+c-b$, $(n-1)a^2+4nb\geqslant0$, $a\geqslant0$ and $b\leqslant\frac{2c}{n}$, where a and b are constants, then M^n is totally umbilical.*

Proof of Theorem 1.8. First we consider the quadratic form

$$D(u,v)=u^2-\frac{n-2}{\sqrt{n-1}}uv-v^2 \tag{3.23}$$

and the orthogonal transformation

$$\begin{aligned}\overline{u}&=\frac{1}{\sqrt{2n}}\left[(1+\sqrt{n-1})u+(1-\sqrt{n-1})v\right],\\ \overline{v}&=\frac{1}{\sqrt{2n}}\left[(\sqrt{n-1}-1)u+(\sqrt{n-1}+1)v\right].\end{aligned} \tag{3.24}$$

Using (3.24), we can rewrite (3.23) as follows

$$\begin{aligned}D(u,v)=D(\overline{u},\overline{v})&=\frac{n}{2\sqrt{n-1}}(\overline{u}^2-\overline{v}^2)\\ &=-\frac{n}{2\sqrt{n-1}}(\overline{u}^2+\overline{v}^2)+\frac{n}{\sqrt{n-1}}\overline{u}^2.\end{aligned} \tag{3.25}$$

From (3.24), we have

$$u^2 + v^2 = \overline{u}^2 + \overline{v}^2. \tag{3.26}$$

Take $u = |\phi|$ and $v = \sqrt{n}|H|$. Substituting u and v into (3.23) and using (3.25), we have

$$\begin{aligned}|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + nc - nH^2 &= nc + D(|\phi|, \sqrt{n}|H|) \\ &= nc + \frac{n}{2\sqrt{n-1}}(\overline{u}^2 - \overline{v}^2) \\ &= nc - \frac{n}{2\sqrt{n-1}}(\overline{u}^2 + \overline{v}^2) + \frac{n}{\sqrt{n-1}}\overline{u}^2.\end{aligned} \tag{3.27}$$

From (3.26), we have $u^2 + v^2 = \overline{u}^2 + \overline{v}^2 = |\phi|^2 + nH^2 = S$. Hence, it follows from (3.27) that

$$|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + nc - nH^2 \geqslant nc - \frac{n}{2\sqrt{n-1}}S. \tag{3.28}$$

If M^n is maximal, i.e., $H \equiv 0$, according to Nishikawa's result [14], we know that M^n is totally geodesic. We can assume that H is not identically zero. By changing the orientation of M^n if necessary, we may assume $\sup H > 0$. Since $S = n^2H^2 + n(n-1)(aH - b)$ in (3.3), combining $\sup S < 2\sqrt{n-1}c$ and $a \geqslant 0$, we can conclude that H is bounded. Hence, by Proposition 3.3 we can obtain a sequence of points $\{x_k\} \in M^n$ such that

$$\lim_{k\to\infty} \sup\left(L(nH)(x_k)\right) \leqslant 0, \qquad \lim_{k\to\infty}(nH)(x_k) = \sup(nH) > 0. \tag{3.29}$$

From (2.10), (3.18) and (3.29), we have

$$\lim_{k\to\infty} S(x_k) = \sup S. \tag{3.30}$$

Combining (3.22), (3.28) and (3.30), we obtain

$$\begin{aligned} 0 &\geqslant \lim_{k\to\infty} \sup\left(L(nH)(x_k)\right) \\ &\geqslant \sup|\phi|^2 \left(\sup|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}\sup|H|\sup|\phi| + nc - n\sup H^2\right) \\ &\geqslant \sup|\phi|^2\left(nc - \frac{n}{2\sqrt{n-1}}\sup S\right).\end{aligned} \tag{3.31}$$

Since $\sup S < 2\sqrt{n-1}c$, we conclude from (3.31) that $\sup|\phi|^2 = 0$. That is, $|\phi|^2 = 0$ which shows M^n is totally umbilical. This completes the proof of Theorem 1.8. $\square$

If L_1^{n+1} is a de Sitter space $\mathbb{S}_1^{n+1}(c)$ in Theorem 1.8, then $-\frac{c_1}{n} = c_2 = c$ and $R = aH + c - b$. Thus, we obtain the following corollary.

Corollary 3.5. *Let $M^n (n \geqslant 3)$ be a complete spacelike hypersurface in a de Sitter space $\mathbb{S}_1^{n+1}(c)$. Suppose that the squared length S of the second fundamental form of M^n satisfies $\sup S < 2\sqrt{n-1}c$. If the normalized scalar curvature R and the mean curvature H of M^n satisfy the following conditions: $R = aH + c - b$, $(n-1)a^2 + 4nb \geqslant 0$ and $a \geqslant 0$, where a and b are constants, then M^n is totally umbilical.*

Remark 3.6. Let $a = 0$ in Corollary 3.5, we know that $R = c - b$ is constant and $R \leqslant c$. Hence, Corollary 3.5 is a generalization of Theorem 1.5.

4 Compact linear Weingarten spacelike hypersurfaces in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$

According to Cheng and Yau $\square$ given in [7], we introduce a self-adjoint operator $\square$ acting on any C^2-function f by

$$\square(f) = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}. \tag{4.1}$$

In order to prove Theorems 1.11 and 1.13, we need the following proposition.

Proposition 4.1. *Let $M^n (n \geqslant 3)$ be a spacelike hypersurface in a locally symmetric Lorentz space L_1^{n+1} satisfying $(*)$. If the normalized scalar curvature R and the mean curvature H of M^n satisfy the following conditions: $R = aH + b_1$, $b_1 = \frac{1}{n(n-1)}\sum_{i,j=1}^n \overline{R}_{ijji} - b$, $(n-1)a^2 + 4nb \geqslant 0$, where a, b and b_1 are constants, then*

$$\square(nH) \geqslant -\frac{1}{2}\Delta\left(n(n-1)R\right) + |\phi|^2 L_{|H|}(|\phi|), \tag{4.2}$$

where $|\phi|^2 = S - nH^2$, $L_{|H|}(|\phi|) = |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + nc - nH^2$, $c = 2c_2 + \frac{c_1}{n} > 0$ and c_1, c_2 are given as in $(*)$.

Proof. Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i\delta_{ij}$. Noticing that $nH\Delta(nH) = \frac{1}{2}\Delta(nH)^2 - n^2|\nabla H|^2$, it follows from (2.3) and (4.1) that

$$\begin{aligned}\square(nH) =& \sum_{i,j}(nH\delta_{ij} - h_{ij})(nH)_{ij} \\ =& \frac{1}{2}\Delta(nH)^2 - n^2|\nabla H|^2 - \sum_i \lambda_i(nH)_{ii} \\ =& -\frac{1}{2}\Delta\left(n(n-1)R\right) + \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_i \lambda_i(nH)_{ii}.\end{aligned} \tag{4.3}$$

Thus, we conclude from (2.12) and (4.3) that

$$\square(nH) \geqslant -\frac{1}{2}\Delta\left(n(n-1)R\right) + \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + |\phi|^2 L_{|H|}(|\phi|), \tag{4.4}$$

where $|\phi|^2 = S - nH^2$ and $L_{|H|}(|\phi|) = |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + nc - nH^2$.

Hence, (4.2) follows from (3.8) and (4.4). Finally, Proposition 4.1 is proved. $\square$

Proof of Theorem 1.11. By using the similar processing as in the proof of Theorem 1.6 on the inequality $L_{\sup|H|}(\sup|\phi|) > 0$, we obtain

$$L_{|H|}(|\phi|) = |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + nc - nH^2 > 0. \tag{4.5}$$

Since M^n is compact and $\square$ is self-adjoint operator, we get

$$\int_{M^n} \square(nH) dv_{M^n} = 0. \tag{4.6}$$

From (4.2) and (4.6), we get

$$0 \geqslant \int_{M^n} |\phi|^2 L_{|H|}(|\phi|) dv_{M^n}, \tag{4.7}$$

where $|\phi|^2 = S - nH^2$ and $L_{|H|}(|\phi|) = |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| + nc - nH^2$.

Hence, we can conclude from (4.5) and (4.7) that $|\phi|^2 = 0$ which shows M^n is totally umbilical. This completes the proof of Theorem 1.11. $\square$

When L_1^{n+1} is a de Sitter space $\mathbb{S}_1^{n+1}(c)$ in Theorem 1.11, we know that $-\frac{c_1}{n} = c_2 = c$ and $R = aH + c - b$. Thus, we obtain the following corollary.

Corollary 4.2. *Let $M^n(n \geqslant 3)$ be a compact spacelike hypersurface in a de Sitter space $\mathbb{S}_1^{n+1}(c)$. If the normalized scalar curvature R and the mean curvature H of M^n satisfy the following conditions: $R = aH + c - b$, $(n-1)a^2 + 4nb \geqslant 0$, $a \geqslant 0$ and $b \leqslant \frac{2c}{n}$, where a and b are constants, then M^n is totally umbilical.*

Remark 4.3. When we take $a = 0$ in Corollary 4.2, we obtain that $R = c - b$ is constant and $\frac{n-2}{n}c \leqslant R \leqslant c$. Thus, Corollary 4.2 is a generalization of Theorem 1.9.

Proof of Theorem 1.13. From (3.28) and (4.7), we obtain

$$0 \geqslant \int_{M^n} |\phi|^2 \left(nc - \frac{n}{2\sqrt{n-1}} S \right) dv_{M^n}. \tag{4.8}$$

Since $S < 2\sqrt{n-1}c$, we can conclude from (4.8) that $|\phi|^2 = 0$ which shows M^n is totally umbilical. This completes the proof of Theorem 1.13. $\square$

When L_1^{n+1} is a de Sitter space $\mathbb{S}_1^{n+1}(c)$ in Theorem 1.13, we know that $-\frac{c_1}{n} = c_2 = c$ and $R = aH + c - b$. Thus, we obtain the following corollary.

Corollary 4.4. *Let $M^n(n \geqslant 3)$ be a compact spacelike hypersurface in a de Sitter space $\mathbb{S}_1^{n+1}(c)$. Suppose that the squared length S of the second fundamental form of M^n satisfies $S < 2\sqrt{n-1}c$. If the normalized scalar curvature R and the mean curvature H of M^n satisfy the following conditions: $R = aH + c - b$ and $(n-1)a^2 + 4nb \geqslant 0$ and $a \geqslant 0$, where a and b are constants, then M^n is totally umbilical.*

When we take $a = 0$ in Corollary 4.4, we obtain that $R = c - b$ is constant and $R \leqslant c$. Thus, we obtain the following corollary.

Corollary 4.5. *Let $M^n(n \geqslant 3)$ be a compact spacelike hypersurface in a de Sitter space $\mathbb{S}_1^{n+1}(c)$ with constant normalized scalar curvature R, $R \leqslant c$. If the squared length S of the second fundamental form of M^n satisfies $S < 2\sqrt{n-1}c$, then M^n is totally umbilical.*

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